

# Black Hole Entropy in Loop Quantum Gravity, Analytic Continuation, and Dual Holography

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A new approach to black hole thermodynamics is proposed in Loop Quantum Gravity (LQG), by defining a new black hole partition function, followed by analytic continuations of Barbero-Immirzi parameter to  $\gamma \in i\mathbb{R}$  and Chern-Simons level to  $k \in i\mathbb{R}$ . The analytic continued partition function has remarkable features: The black hole entropy  $S = A/4\ell_P^2$  is reproduced correctly for infinitely many  $\gamma = i\eta$ , at least for  $\eta \in \mathbb{Z} \setminus \{0\}$ . The near-horizon Unruh temperature emerges as the pole of partition function. Interestingly, by analytic continuation the partition function can have a dual statistical interpretation corresponding to a dual quantum theory of  $\gamma \in i\mathbb{Z}$ . The dual quantum theory implies a semiclassical area spectrum for  $\gamma \in i\mathbb{Z}$ . It also implies that at a given near horizon (quantum) geometry, the number of quantum states inside horizon is bounded by a holographic degeneracy  $d = e^{A/4\ell_P^2}$ , which produces the Bekenstein bound from LQG. The result in [1] also receives a justification here.

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It is well-known that black hole, as a system arise from General Relativity (GR), has remarkable thermodynamical properties [2]. In particular, black hole has an entropy proportional to its area by  $S = A/4\ell_P^2$ . The black hole entropy results in important ramifications such as the Bekenstein's entropy bound, and the covariant Bousso's bound [3], which conjectures that the number of microstates inside a (spatial) region is bounded by  $e^{A/4\ell_P^2}$  where  $A$  is the area surrounding the region. The conjecture leads to the holographic principle for quantum gravity [4].

The statistical origin of black hole entropy needs to be explained by quantum gravity. In this paper we propose a new approach to black hole entropy in Loop Quantum Gravity (LQG) [5]. There has been a long history of computing black hole entropy from LQG (e.g. [6–8]). The resulting black hole entropy has had a famous dependence of Barbero-Immirzi parameter  $\gamma \in \mathbb{R}$ . Reproducing  $S = A/4\ell_P^2$  relies on fine-tuning  $\gamma$  to a single critical value  $\gamma_0$ . The situation is improved by the recent progress [9, 10], where an area-energy relation  $E = \frac{A}{8\pi\ell}$  allows to equivalently formulate black hole as a (grand) canonical ensemble. However it still has not been clear yet how exactly  $A/4\ell_P^2$  emerges as black hole entropy from LQG framework.

In this work, a new grand canonical partition function  $\mathcal{Z}$  is proposed for LQG black hole. Then we analytic continue the partition function to purely imaginary Barbero-Immirzi parameter  $\gamma \in i\mathbb{R}$  (up to a small real part). Correspondingly, the Chern-Simons level is complexified  $k \in i\mathbb{R}$ , motivated by a relation between  $k$  and  $\gamma$  in isolated horizon context [7]. Motivated by [1], we take the viewpoint that an object of LQG with complex  $\gamma$  is defined by the corresponding object from well-defined quantization with real  $\gamma$ , followed by an analytic continuation of  $\gamma$  to complex plane. Interestingly, the analytic continuation results in the following remarkable features:

- The analytic continued black hole partition function  $\mathcal{Z}$  reproduces correctly the entropy  $S = A/4\ell_P^2$  as the leading contribution, supplemented by quantum and UV corrections.

- The derivation works at least for  $\gamma \simeq i\eta$  ( $\eta \in \mathbb{Z} \setminus \{0\}$ ) up to small real part. There are infinitely many allowed purely imaginary  $\gamma$ , all resulting in  $S = A/4\ell_P^2$ . The case of Ashtekar's variables [11] is included as  $\gamma = \pm i$ . Generalization to noninteger  $\eta$  may rely on a technical assumption of analytic continuation.
- The Unruh temperature  $\beta_U = \frac{2\pi\ell}{\ell_P}$  (of near horizon observer with distance  $\ell$ ) appears as a pole in the analytic continued partition function. The naturality of  $\beta_U$  is also suggested by [12] from a different point of view.
- Close to Unruh temperature,  $\mathcal{Z}$  can have a *dual* interpretation as a statistical system, corresponding to a *dual quantum theory* associated with  $\gamma = i\eta$ . The resulting dual quantum theory has a (semiclassical) area spectrum  $A = 8\pi|\eta|\ell_P^2 \sum_l s_l$  ( $s_l \in \mathbb{R}_+$ ) up to a specific rescaling.
- More importantly, in the dual quantum theory, at a given near horizon (quantum) geometry, the number of quantum states inside horizon is bounded by the degeneracy  $d \simeq e^{A/4\ell_P^2}$ . Such a holographic degeneracy produces the Bekenstein bound from LQG. The assumption of holographic degeneracy in [13] also receives a justification here.

On the other hand, the positivity of black hole energy spectrum, the analyticity (holomorphicity), and the existence of dual statistical interpretation of  $\mathcal{Z}$ , suggests a specific 1st order quantum correction to the classical energy-area relation proposed in [9]. The correction may come from the radiative correction from LQG [14].

Black hole in LQG is described in terms of an  $SU(2)$  Chern-Simons theory with level  $k$  [7]

$$S_{CS}[\mathcal{A}] = \frac{k}{4\pi} \int_H \text{tr} \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \quad (1)$$

where  $H$  is the black hole horizon with spatial area  $A$ . The Chern-Simons level  $k$  will be complexified to  $k \in i\mathbb{R}$  as analytic continuing  $\gamma \in i\mathbb{R}$ .

The near-horizon quantum geometry of black hole are described by  $N$  punctures on spatial section of  $H$  from Wilson lines in Chern-Simons theory, with a set of spins/areas  $\{j_l\}_{l=1}^N$  [7, 15]. Given  $\{j_l\}_{l=1}^N$ , A quantum state inside horizon is a Chern-Simons state on  $S^2$  with  $N$  punctures colored by  $\{j_l\}_{l=1}^N$ . The Hilbert space has the dimension given by the famous Verlinde formula [16, 17] ( $d_l = 2j_l + 1$ ):

$$\dim_k(\vec{j}) = \frac{2}{k+2} \sum_{d=1}^{k+1} \sin^2\left(\frac{\pi d}{k+2}\right) \prod_{l=1}^N \frac{\sin\left(\frac{\pi d d_l}{k+2}\right)}{\sin\left(\frac{\pi d}{k+2}\right)} \quad (2)$$

which is the degeneracy the black hole microstates at a given near-horizon geometry. The LQG approach of black hole entropy has been based on the Verlinde formula, which has led to the well-known  $\gamma$ -dependence [8]. Recently there has been an interesting observation from [1]: if  $\dim_k(\vec{j})$  are analytic continued to  $j_l = is_l - 1/2$ , its asymptotic behavior as  $s_l$  large gives  $e^{A/4\ell_P}$ , in terms of a conjectured LQG area spectrum when  $\gamma = \pm i$ . The result motivates that the  $A/4\ell_P$ -law may naturally come from a quantum theory with complex Ashtekar connection with purely imaginary  $\gamma$ . Such viewpoint motivates the work here and has been adopted in several recently works [18]. However such an interesting result in [1] is mysterious and has to be justified. When  $j$  is complexified, the Verlinde formula loses the meaning as a Hilbert space dimension. It has not been clear yet if the result in [1] counts the quantum states of any system. Such an issue in [1] will be justified in the following analysis.

Let's consider a quantum black hole horizon described by a gas of  $N$  punctures. Motivated by [9, 13], a canonical partition function is defined by summing over spin configurations, with a degeneracy factor given by  $\dim_k(\vec{j})$ :

$$Z_N = \frac{1}{N!} \sum_{j_1 \dots j_N = \frac{1}{2}}^{k/2} \dim_k(\vec{j}) e^{-\beta E(\vec{j})} \quad (3)$$

The grand canonical partition function is defined by  $\mathcal{Z} = \sum_N Z_N e^{\mu N}$ . Here  $\dim_k(\vec{j})$  is a faithful counting of degenerate states with a given set of  $\{j_l\}_l$ .  $1/N!$  is a Gibbs factor of indistinguishable punctures. The Hamiltonian is defined by

$$E = \gamma \frac{\ell_P^2}{\ell} \sum_{l=1}^N \left[ j_l + \frac{1}{2} + f(\gamma, k) \right]. \quad (4)$$

In the semiclassical large- $j$  regime, the energy spectrum proposed here is consistent with the LQG area  $A = 8\pi\gamma\ell_P^2 \sum_{l=1}^N \sqrt{j_l(j_l + 1)}$  and the classical energy-area relation  $E = \frac{A}{8\pi\ell}$  of near-horizon observer [9, 10, 13].  $\ell$  is the small proper distance from the horizon.  $f(\gamma, k)$  stands for a possible quantum deviation from the classical area-energy relation. It has to be a holomorphic function in order to perform analytic continuation. It has to be real as  $\gamma, k \in \mathbb{R}$  for a Hermitian Hamiltonian. Our analysis will fix  $f(\gamma, k)$  to the following form:

$$f(\gamma, k) = \frac{1}{2\pi\gamma} [m \log k + \log \alpha_m(\gamma)] \quad (5)$$

with parameters  $m \geq 0$  and  $\alpha_m(\gamma) > 0$  satisfying certain condition. The  $\log k$  term may relates to the self-energy from spin-foam amplitude [14].

Here we have analytic continued  $Z_N$  to complex  $\gamma$ -plane, and set  $\gamma = -i\eta$ , where  $\eta = \eta_0 - i\varepsilon$  ( $\varepsilon$  small) with  $\eta_0 \in \mathbb{Z} \setminus \{0\}$ . Without loss of generality, we set  $\eta_0 > 0$  in the main content. Our following analysis is symmetric under  $\eta \rightarrow -\eta$ .

The local temperature of the near-horizon observer is the Unruh temperature  $\beta_U = \frac{2\pi\ell}{\ell_P^2}$ . The range of sum  $\sum_j$  in Eq.(3) is from  $\frac{1}{2}$  to  $\frac{k}{2}$ , i.e. the integrable representations in  $SU(2)_k$  affine Lie algebra [19].

Now the summand in  $Z_N$  becomes oscillatory, which would make  $Z_N$  lose the interpretation as a statistical partition function. However the following procedure leads to a “dual statistical system”, which does interpret  $Z_N$  as a statistical partition function. Insert the Verlinde formula and sum over  $j_l$ ,

$$Z_N = c_{N,k} \sum_{d=1}^{k+1} \sin^{2-N}\left(\frac{\pi d}{k+2}\right) \prod_{l=1}^N \sum_{d_l=2}^{k+1} \left[ e^{i\Delta_d^+ \frac{d_l}{2}} - e^{i\Delta_d^- \frac{d_l}{2}} \right], \quad (6)$$

where  $d_l = 2j_l + 1$  and

$$\Delta_d^\pm = \eta\beta \frac{\ell_P^2}{\ell} \pm \frac{2\pi d}{k+2}, \quad c_{N,k} = \frac{(-i)^N}{N!} \frac{2^{1-N}}{k+2} e^{Nf(-i\eta, k) i\eta\beta \frac{\ell_P^2}{\ell}}. \quad (7)$$

The sum  $\sum_{d_l=2}^{k+1}$  can be performed easily. Then  $Z_N$  reads

$$c_{N,k} \sum_{d=1}^{k+1} \sin^{2-N}\left(\frac{\pi d}{k+2}\right) \left[ \frac{e^{i\Delta_d^+} \left( e^{\frac{\pi}{2}\Delta_d^+} - 1 \right)}{e^{\frac{1}{2}\Delta_d^+} - 1} - \frac{e^{i\Delta_d^-} \left( e^{\frac{\pi}{2}\Delta_d^-} - 1 \right)}{e^{\frac{1}{2}\Delta_d^-} - 1} \right]^N.$$

Now we complexify the Chern-Simons level  $k = i\lambda - 2$  in the partition function, where  $\lambda \in \mathbb{R}_+$  is large but finite. There is an obvious difficulty that  $k$  appears as the upper bound of the sum  $\sum_{d=1}^{k+1}$ . However no one prevents us to firstly make the replacement  $k = i\lambda - 2$  for  $k$  appearing inside the summand. After replacement  $Z_N$  reads

$$c_{N,\lambda} \sum_{d=1}^{k+1} \sin^{2-N}\left(\frac{\pi d}{i\lambda}\right) \left[ \frac{e^{i\Delta_d^+} \left( e^{\frac{-i\lambda-2}{2}\Delta_d^+} - 1 \right)}{e^{\frac{1}{2}\Delta_d^+} - 1} - \frac{e^{i\Delta_d^-} \left( e^{\frac{-i\lambda-2}{2}\Delta_d^-} - 1 \right)}{e^{\frac{1}{2}\Delta_d^-} - 1} \right]^N.$$

$k$  appearing at  $\sum_{d=1}^{k+1}$  is temporarily kept unchanged. One should firstly perform the sum then analytic continue  $k$ . Here  $\Delta_d^\pm$  and  $c_{N,\lambda}$  read

$$\Delta_d^\pm = \eta\beta \frac{\ell_P^2}{\ell} \pm \frac{2\pi d}{i\lambda}, \quad c_{N,\lambda} = \frac{(-i)^N}{N!} \frac{2^{1-N}}{i\lambda} e^{Nf(-i\eta, i\lambda) i\eta\beta \frac{\ell_P^2}{\ell}}. \quad (8)$$

The partition function has a series of nontrivial poles at

$$\Delta_d^\pm = 4\pi q_\pm, \quad (q \in \mathbb{Z}, q \neq 0) \quad (9)$$

As long as  $q \neq 0$ , the residue in each factor of summand at the pole is nonzero, thanks to the complexification of  $k$ .

We firstly consider the case  $\eta_0$  is an odd integer, i.e.  $\eta = 2q - 1 - i\frac{x}{\lambda}$  ( $q, x \in \mathbb{Z}_+, x > 0, x \ll \lambda$ ) with small imaginary

part, it picks the  $k+2-x$  term (close to the top of the sum) outside the sum  $\sum_{d=1}^{k+1}$ , i.e. we write the sum in  $Z_N$  as

$$\left[ \frac{e^{i\Delta_{k+2-x}^+} \left( e^{\frac{-i-2i}{2}\Delta_{k+2-x}^+} - 1 \right)}{e^{\frac{i}{2}\Delta_{k+2-x}^+} - 1} - \frac{e^{i\Delta_{k+2-x}^-} \left( e^{\frac{-i-2i}{2}\Delta_{k+2-x}^-} - 1 \right)}{e^{\frac{i}{2}\Delta_{k+2-x}^-} - 1} \right]^N + \sum_{d \neq k+2-x}^{k+1} \dots$$

The  $k+2-x$  term outside the sum can be analytic continued to  $k = i\lambda - 2$  without difficulty. Then the Unruh temperature  $\beta_U = \frac{2\pi\ell}{\ell_P}$  appears as the pole of this term

$$0 = \Delta_{i\lambda-x}^+ - 4\pi q = \left( 2q - 1 - i\frac{x}{\lambda} \right) \frac{\ell_P^2}{\ell} (\beta - \beta_U) \quad (10)$$

The residue of the pole within the factor is  $2i$  approximately, as we ignore the exponentially decaying  $e^{-\lambda 4\pi q}$ .

Such a pole can never appear from the rest of terms in  $\sum_{d \neq k+2-x}^{k+1}$ , it also doesn't coincide with the pole given by  $\Delta_{i\lambda-x}^-$ . Indeed, if we pick out the  $d$  term and analytic continue in the same way as above, close to  $\beta_U$

$$\Delta_d^\pm = (\Delta_{i\lambda-x}^\pm - 4\pi q) + \frac{2\pi(x \pm d - i\lambda)}{i\lambda} + 4\pi q \quad (11)$$

If  $\Delta_d^\pm = 4\pi m$  with  $m \in \mathbb{Z}$  when  $\Delta_{i\lambda-x}^\pm = 4\pi q$ ,  $\frac{(x \pm d - i\lambda)}{i\lambda}$  would be an even number, which implies  $d = \pm(2m+1)i\lambda \mp x$  after complexification. It can only happen in the  $\Delta_d^+$  case with  $d = i\lambda - x$  since originally  $0 < d < k+2$ . However it can nevertheless happen that  $\Delta_d^\pm = 4\pi\mathbb{Z} + o(\frac{1}{\lambda})$  at  $\beta_U$ , e.g. modulo  $4\pi\mathbb{Z}$ ,  $\Delta_{i\lambda-x}^-(\beta_U) = \frac{2\pi}{i\lambda} - 4\pi$  and  $\Delta_{i\lambda-x+1}^+(\beta_U) = \frac{1}{i\lambda}$ , i.e. other terms with  $d \neq k+2-x$  can have contribution of  $o(\lambda)$ .

The next task is to show the sum  $\sum_{d \neq k+2-x}^{k+1}$  is negligible if  $\beta$  is sufficiently close to  $\beta_U$ . We may estimate the sum by an integral up to  $o(1/k)$  i.e. we write the sum to be

$$(k+2) \left[ \int_{\frac{1}{k+2}}^{\frac{k+2-x-1}{k+2}} d \left( \frac{d}{k+2} \right) \dots + \int_{\frac{k+2-x+1}{k+2}}^{\frac{k+1}{k+2}} d \left( \frac{d}{k+2} \right) \dots \right] \quad (12)$$

Analytic continuation  $k = i\lambda - 2$  corresponds a rotation of integration contour ( $\xi = d/\lambda$ ):

$$\lambda \left[ \int_{\frac{1}{\lambda}}^{\frac{i\lambda-x-1}{\lambda}} d\xi \dots + \int_{\frac{i\lambda-x+1}{\lambda}}^{\frac{i\lambda-1}{\lambda}} d\xi \dots \right] \quad (13)$$

where the integrand reads

$$\frac{1}{\sin^{N-2}(-i\pi\xi)} \left[ \frac{e^{i\Delta_\xi^+} \left( e^{\frac{-i-2i}{2}\Delta_\xi^+} - 1 \right)}{e^{\frac{i}{2}\Delta_\xi^+} - 1} - \frac{e^{i\Delta_\xi^-} \left( e^{\frac{-i-2i}{2}\Delta_\xi^-} - 1 \right)}{e^{\frac{i}{2}\Delta_\xi^-} - 1} \right]^N \quad (14)$$

where  $\Delta_\xi^\pm = \eta\beta\frac{\ell_P^2}{\ell} \pm 2\pi(-i)\xi$ . By above discussion, when  $\beta$  close to  $\beta_U$ , the integrand has a  $N$ -th order pole at  $\xi = \frac{i\lambda-x}{\lambda}$ . When  $N > 2$  it has additional  $(N-2)$ -th order pole at  $\xi = 0, i$ . However all the poles have a  $1/\lambda$ -distance away from the integration contour. Since the pole  $\xi = \frac{i\lambda-x}{\lambda}$  is close to  $\xi = i$ , the integral Eq.(13) grows as  $\lambda^{2N-2}$ , which is also the leading behavior of the sum  $\sum_{d \neq k+2-x}^{k+1}$  after analytic continuation. On the other hand, the  $d = k+2-x$  term outside the sum is of

the order  $\lambda^{N-2}\delta_\beta^{-N}$ . Therefore, when we are inside the regime that  $\delta_\beta = \eta\frac{\ell_P^2}{\ell}(\beta - \beta_U) \ll \frac{1}{\lambda}$ , the contribution from  $\sum_{d \neq k+2-x}^{k+1}$  is negligible for all  $N$ .

The partition function is simplified dramatically after the approximation. As  $\lambda \gg 1$

$$Z_N \simeq \frac{1}{N!} \left[ \frac{2\pi^2 x^2}{(i\lambda)^3} \right] i^N e^{Nf(-i\eta, i\lambda)2\pi i\eta} \left( \frac{\lambda}{\pi x} \right)^N \left[ \frac{\ell}{\eta\ell_P^2(\beta - \beta_U)} \right]^N \quad (15)$$

If  $\eta_0$  is an even integer, i.e.  $\eta = 2q + i\frac{x}{\lambda}$  ( $q \in \mathbb{Z}_+$ ,  $x > 0$ ,  $x \ll \lambda$ ), it picks up the poles close to the bottom of the sum  $\sum_{d=1}^{k+1}$ , i.e. the term with  $d = x \in \mathbb{Z}_+$  with

$$0 = \Delta_x^+ - 4\pi q = \left( 2q + i\frac{x}{\lambda} \right) \frac{\ell_P^2}{\ell} (\beta - \beta_U) \quad (16)$$

The estimate can be carried out in the same way as above, by replacement  $x \rightarrow i\lambda - x$ . A similar result holds, i.e. as  $\eta = 2q + i\frac{x}{\lambda}$ , the term with  $d = x$  is picked up as the leading contribution, as long as  $\delta_\beta \ll \frac{1}{\lambda}$ . The resulting partition function is exactly the same as Eq.(15).

The derivation with integer  $\eta_0$  works because it is allowed to pick up terms at the top or bottom in the sum  $\sum_{d=1}^{k+1}$  for analytic continuation of  $k$ . It may or may not work for terms in the middle. e.g. If we assume picking up  $d = \frac{1}{2}(k+1)$  is allowed, the above derivation generalizes to noninteger  $\eta_0$ . However  $\frac{1}{2}(k+1)$  may not always a integer for all  $k$ , the term  $d = \frac{1}{2}(k+1)$  may or may not appear in the sum. So generalization to noninteger  $\eta_0$  may rely on nontrivial assumptions.

In Eq.(15),  $\left[ \frac{2\pi^2 x^2}{(i\lambda)^3} \right]$  only contributes the logarithmic correction in grand potential  $\log \mathcal{Z}$ . The rest part in  $Z_N$  has to be real and positive in order to have a dual statistical interpretation. Thus  $e^{f(-i\eta, i\lambda)2\pi i\eta} = \chi(-i\eta, i\lambda)$ , where both  $f$  and  $\chi$  are holomorphic in  $\lambda, \eta$  and  $\chi(-i\eta, i\lambda) \in i\mathbb{R}_-$ . As  $f, \chi$  are holomorphic, this equation holds on the whole complex plane, which implies  $e^{-f(\gamma, k)2\pi\gamma} = \chi(\gamma, k)$ .  $f(\gamma, k)$  is real for a Hermitian Hamiltonian  $E$ , which implies  $\chi(\gamma, k) \in \mathbb{R}_+$ . Expand  $\chi$  into power series  $\chi(\gamma, k) = \sum_m k^m \alpha_m(\gamma)$ , and keep only the leading term as  $k$  large. If the leading order would be of  $o(k^{m>0})$ , it would give  $f(\gamma, k) = \frac{-m}{2\pi\gamma} \log k$  as the leading order, which would produce negative  $E$  for small spins. A positive definite energy spectrum implies the leading order of  $\chi(\gamma, k)$  is  $k^{-m} \alpha_m(\gamma)$  with  $m \geq 0$ .  $\alpha_m(\gamma)$  should satisfy  $\alpha_m(\gamma) \in \mathbb{R}_+$  and  $i^{-m+1} \alpha_m(-i\eta) \in \mathbb{R}_+$ . So we fix  $f(\gamma, k)$  to the form in Eq.(5).

Here we allow the creation and annihilation of the punctures on the horizon (or a sum over graphs in LQG terminology). We define a grand canonical partition function  $\mathcal{Z} = \sum_N Z_N e^{\mu N}$  where  $\mu$  is a postulated chemical potential.

$$\log \mathcal{Z} \simeq \frac{\lambda|\chi|}{\pi x} e^\mu \frac{\ell}{\eta\ell_P^2(\beta - \beta_U)} - 3 \log \lambda. \quad (17)$$

The leading contribution to mean energy  $U = -\partial_\beta \log \mathcal{Z}$  can be computed straightforwardly as  $k$  being large:

$$U[\beta_-] \simeq \frac{\lambda|\chi|}{\pi x} e^\mu \frac{\ell}{\eta\ell_P^2(\beta - \beta_U)^2} \left[ 1 + o(\lambda^{-1}) \right] \quad (18)$$

which relates the horizon area by the classical relation  $U = \frac{A}{8\pi\ell}$ . If  $m = 0$  in Eq.(5) then  $\chi \sim o(1)$ , we obtain the relation  $\eta \frac{\ell_p^2}{\ell} (\beta - \beta_U) \equiv \delta_\beta \propto \sqrt{\lambda \ell_p^2 / A}$ . If  $m = 1$  then  $\chi \sim o(1/\lambda)$  and  $\delta_\beta \propto \sqrt{\ell_p^2 / A}$ .  $\delta_\beta$  becomes finer as  $m$  increase.

The entropy from the grand canonical ensemble is given by  $S = \beta U + \log \mathcal{Z}$ . The leading contribution of entropy is given by  $\beta U$  because  $\log \mathcal{Z} \sim \delta_\beta^{-1}$  while  $U \sim \delta_\beta^{-2}$ . Therefore the leading contribution to the entropy at  $\beta_U$  is given by

$$S = \frac{A}{4\ell_p^2} [1 + o(\lambda^{-1}) + o(\delta_\beta)] - 3 \log \lambda,$$

which reproduces the classical law  $S = A_H / 4\ell_p^2$  up to LQG corrections for infinitely many  $\gamma = -\mathbb{Z}_+ i + \varepsilon$ .

Before the analytic continuation, Chern-Simons level  $k$  stands for the maximal area allowed at a single puncture (defect) on the horizon. The area of a single puncture should not be too large, otherwise it would break the macroscopic smoothness of the horizon. The situation is similar to the case of spinfoam LQG [20], where the spin should be cut-off by introducing quantum group or Chern-Simons theory [21, 22]. The spin cut-off should not be too large, in order to preserve the macroscopic smoothness. Here  $\lambda$  is assumed of the same scale as  $k$ . For example, if the spins are cut-off at the Grand Unification Scale,  $k\ell_p^2$  or  $\lambda\ell_p^2$  is the area scale of GUT, i.e.  $k, \lambda \sim 10^6$ . The Schwarzschild horizon area of the sun is  $A_H \sim 10^6 m^2$ . The maximal  $\delta_\beta \propto \sqrt{\lambda \ell_p^2 / A} \sim 10^{-35}$  is a tiny LQG correction. This example also illustrates our approximation scheme  $\delta_\beta \ll 1/\lambda$  is natural.

As an analog of covariant LQG [20],  $o(1/\lambda)$  or  $o(1/k)$  are the quantum corrections relating to the large- $j$  expansion near the cut-off, while  $o(\delta_\beta)$  are high curvature UV corrections since  $A$  relates to the curvature radius. The analysis here is valid in a semiclassical low energy regime  $\ell_p^2 \ll k\ell_p^2 \ll A$ . It is consistent with the proposal in [13].

Interestingly there exists a dual statistical system emerges from the partition function  $Z_N$  by the above analysis, although its expression Eq.3 loses the obvious statistical interpretation as  $\gamma = -i\eta = -i\eta_0 + \varepsilon$ ,  $\eta_0 \in \mathbb{Z}_+$ . As  $\beta \rightarrow \beta_U$  from  $\beta > \beta_U$ , the leading contribution to  $Z_N$  in Eq.(15), which is responsible for the leading energy and entropy, can be written as an integral up to prefactor that becomes logarithmic corrections in  $\log \mathcal{Z}$ ,

$$Z_N \propto \frac{1}{N!} \int_{\mathbb{R}_+^N} d^N s \prod_{l=1}^N e^{2\pi\eta\zeta s_l - \beta\eta\frac{\ell_p^2}{\ell}\zeta s_l}, \quad \zeta = \frac{\pi x}{\lambda|\chi|} > 0 \quad (19)$$

which interprets  $Z_N$  as a statistical system with continuous energy spectrum  $E = \eta_0 \frac{\ell_p^2}{\ell} \zeta \sum_{l=1}^N s_l$  ( $s_l > 0$ ) and degeneracy  $d(\vec{s}) = e^{2\pi\eta_0 \zeta \sum_{l=1}^N s_l}$ . It implies that by analytic continuation, there exists a dual quantum theory of LQG with  $\gamma = -i\mathbb{Z}$ , which has a semiclassically continuous area spectrum  $A = 8\pi\eta_0 \ell_p^2 \zeta \sum_{l=1}^N s_l$  by  $E = \frac{A}{8\pi\ell}$ . The near-horizon quantum geometry is described in dual quantum theory by the number  $N$  of punctures and a set of dual quantum areas  $\{s_l\}_{l=1}^N$ .

Then importantly, the degeneracy of the dual quantum system is holographic, by

$$\log d(\vec{s}) = \frac{A}{4\ell_p^2}, \quad (20)$$

which shows that the maximal number of black hole microstates of a given near-horizon quantum geometry  $\{s_l\}_{l=1}^N$  is given by the Bekenstein bound.

In the case of Ashtekar variable with  $\eta_0 = 1$ , and if one takes  $x = 1$ ,  $|\chi| = \frac{\pi}{\lambda}$  ( $m = 1$  in Eq.(5)), the degeneracy in the dual system Eq.(19) reduces to  $d(\vec{s}) = e^{2\pi \sum_{l=1}^N s_l}$ , whose origin is exactly the factor  $\prod_l \sin \frac{\pi d d_l}{k+2}$  in highest term  $d = k+1$  in the Verlinde formula Eq.(2). In [1], by complexifying the spins  $j = is - \frac{1}{2}$  and take  $s, k$  to be large,  $d = k+1$  term is picked up as the leading order, and the factor  $\prod_l \sin \frac{\pi d d_l}{k+2}$  transforms into  $e^{2\pi \sum_{l=1}^N s_l}$ . It has not been clarified in [1] if  $e^{2\pi \sum_{l=1}^N s_l}$  counts the quantum states of any system. However, from the above analysis, the result from [1] is justified as a state-counting in the dual quantum theory in the special case  $\eta_0 = 1$ . Furthermore the assumption of holographic degeneracy in [13] also receives a justification here.

The dual statistical system Eq.(19) or  $\mathcal{Z}$  can be understood as  $\int Dg^{(2)} \exp[-(\beta \frac{\ell_p^2}{\ell} - 2\pi) \frac{A[g^{(2)}]}{8\pi\ell_p^2}]$ .  $g^{(2)}$  denotes a metric on the near-horizon 2-surface. It's consistent with an Euclidean path integral of Einstein gravity with a conical deficit angle  $2\pi - \beta \frac{\ell_p^2}{\ell}$  at the horizon [23, 24]. It justifies the argument in [13] which based on the assumption of holographic degeneracy. It also suggests that there should be a derivation of Eq.(19) from covariant LQG via semiclassical low energy approximation, given that covariant LQG reproduces Einstein gravity in the semiclassical low energy regime [20, 25]. Such a top-down approach to black hole thermodynamics is a research undergoing.

Finally, we remark that although the above derivation is for  $\eta_0 > 0$ , the generalization to  $\eta_0 < 0$  ( $k = -i\lambda$ ,  $\lambda > 0$  correspondingly) is straightforward, and only amounts to generalize the dual area spectrum by  $A = 8\pi|\eta_0\zeta|\ell_p^2 \sum_{l=1}^N s_l$  and the holographic degeneracy by  $\log d(\vec{s}) = 2\pi|\eta_0\zeta| \sum_{l=1}^N s_l$ . All the above results are valid to all  $\eta_0 \in \mathbb{Z} \setminus \{0\}$ .

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